

ON DEFORMATION WITH CONSTANT MILNOR NUMBER AND NEWTON POLYHEDRON

OULD M ABDERRAHMANE

ABSTRACT. We show that every μ -constant family of isolated hypersurface singularities satisfying a nondegeneracy condition in the sense of Kouchnirenko, is topologically trivial, also is equimultiple.

Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated singularity. The Milnor number of a germ f , denoted by $\mu(f)$, is algebraically defined as the $\dim \mathcal{O}_n/J(f)$, where \mathcal{O}_n is the ring of complex analytic function germs $: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ and $J(f)$ is the Jacobian ideal in \mathcal{O}_n generated by the partial derivatives $\{\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\}$. We recall that the multiplicity $m(f)$ is defined as the lowest degree in the power series expansion of f at $0 \in \mathbb{C}^n$. Let $F: (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be the deformation of f given by $F(z, t) = f(z) + \sum c_\nu(t)z^\nu$, where $c_\nu: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ are germs of holomorphic functions. We use the notation $F_t(z) = F(z, t)$ when t is fixed. Let m_t denote the multiplicity and μ_t denote the Milnor number of F_t at the origin. The deformation F is equimultiple (resp. μ -constant) if $m_0 = m_t$ (resp. $\mu_0 = \mu_t$) for small t . It is well-known by the result of Lê-Ramanujam [9], that for $n \neq 3$, the topological type of the family F_t is constant under μ -constant deformations. The question is still open for $n = 3$. However, under some additional assumption, positive answers have been given. For example, if F_t is non-degenerate in the sense of Kouchnirenko [7] and the Newton boundary $\Gamma(F_t)$ of F_t is independent of t , i.e., $\Gamma(F_t) = \Gamma(f)$, it follows that $\mu^*(F_t)$ is constant, and hence F_t is topologically trivial (see [12, 16] for details). Motivated by the Briançon-Speder μ -constant family $F_t(z) = z_1^5 + z_2 z_3^7 + z_2^{15} + t z_1 z_3^6$, which is topologically trivial but not μ^* -constant, M. Oka [13] shows that any non-degenerate family of type $F(z, t) = f(z) + t z^A$ for $A = (A_1, \dots, A_n) \in \mathbb{N}^n$, where \mathbb{N} is the set of nonnegative integers and $z^A = z_1^{A_1} z_2^{A_2} \dots z_n^{A_n}$ as usual, is topologically trivial, under the assumption of μ -constancy. Our purpose of this paper is to generalize this result, more precisely, we show that every μ -constant non-degenerate family F_t with not necessarily Newton boundary $\Gamma(F_t)$ independent of t , is topologically trivial. Moreover, we show that F is equimultiple, which gives a positive answer to a question of Zariski [19, 5, 14] for a non-degenerate family. To prove the main result (Theorem 1.1 below), we shall use the notion of (c) -regularity in stratification theory, introduced by K. Bekka in [3], which is weaker than Whitney regularity, nevertheless (c) -regularity implies topological triviality. First, we give a characterization of (c) -regularity (Theorem 2.1 below). By using it, we can show that the μ -constancy condition for a non-degenerate family implies Bekkas (c) -regularity condition and then obtain the topological triviality as a corollary.

2010 *Mathematics Subject Classification.* Primary 14B05, 32S05.

This research was supported by the Japan Society for the Promotion of Science.

Notation. To simplify the notation, we will adopt the following conventions : for a function $F(z, t)$ we denote by ∂F the gradient of F and by $\partial_z F$ the gradient of F with respect to variables z .

Let $\varphi, \psi: (\mathbb{C}^n, 0) \rightarrow \mathbb{R}$ be two function germs. We say that $\varphi(x) \lesssim \psi(x)$ if there exists a positive constant $C > 0$ and an open neighborhood U of the origin in \mathbb{C}^n such that $\varphi(x) \leq C \psi(x)$, for all $x \in U$. We write $\varphi(x) \sim \psi(x)$ if $\varphi(x) \lesssim \psi(x)$ and $\psi(x) \lesssim \varphi(x)$. Finally, $|\varphi(x)| \ll |\psi(x)|$ (when x tends to x_0) means $\lim_{x \rightarrow x_0} \frac{\varphi(x)}{\psi(x)} = 0$.

1. NEWTON POLYHEDRON, MAIN RESULTS

First we recall some basic notions about the Newton polyhedron (see [7, 12] for details), and state the main result.

Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic function defined by a convergent power series $\sum c_\nu x^\nu$, we define $\text{supp}(f) = \{\nu \in \mathbb{N}^n : c_\nu \neq 0\}$. Also, let $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n, \text{ each } x_i \geq 0, i = 1, \dots, n\}$. The Newton polyhedron of f , denoted by $\Gamma_+(f) \subset \mathbb{R}^n$ is defined by the convex hull of $\{k + v \mid k \in \text{supp}(f), v \in \mathbb{R}_+^n\}$, and let $\Gamma(f)$ be the Newton boundary, i.e., the union of the compact faces of $\Gamma_+(f)$. For a face γ of $\Gamma(f)$, we write $f_\gamma(z) := \sum_{\nu \in \gamma} c_\nu x^\nu$. We say that f is non-degenerate if, for any face γ of $\Gamma(f)$, the equations $\frac{\partial f_\gamma}{\partial x_1} = \dots = \frac{\partial f_\gamma}{\partial x_n} = 0$ have no common solution on $x_1 = \dots = x_n \neq 0$. The power series f is said to be convenient if $\Gamma_+(f)$ meets each of the coordinate axes. We let $\Gamma_-(f)$ denote the compact polyhedron which is the cone over $\Gamma(f)$ with the origin as a vertex. When f is convenient, the Newton number $\nu(f)$ is defined as $\nu(f) = n!V_n - (n-1)!V_{n-1} + \dots + (-1)^{n-1}V_1 + (-1)^n$, where the V_n are the n -dimensional volumes of $\Gamma_-(f)$ and for $1 \leq k \leq n-1$, V_k is the sum of the k -dimensional volumes of the intersection of $\Gamma_-(f)$ with the coordinate planes of dimension k . The Newton number may also be defined for a non-convenient analytic function (see [6]). Finally, we define the Newton vertices of f as $\text{ver}(f) = \{\alpha : \alpha \text{ is a vertex of } \Gamma(f)\}$.

Now we can state the main result

Theorem 1.1. *Let $F: (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that F_t is non-degenerate. Then F_t is topologically trivial, and moreover, F is equimultiple.*

Remark 1.2. In the above theorem, we do not require the independence of t for the Newton boundary $\Gamma(F_t)$

2. CRITERION FOR (c)-REGULARITY

By way of notation, we let $G(k, n)$ denote the set of k -dimensional linear subspace of the vector space \mathbb{K}^n , where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

Let M be a smooth manifold, and let X, Y be smooth submanifolds of M such that $Y \subset \overline{X}$ and $X \cap Y = \emptyset$.

(i) (Whitney (a) -regularity)

(X, Y) is (a) -regular at $y_0 \in Y$ if :

for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{T_{x_i}X\}$ tends to τ in the grassmannian $G(\dim X, \dim M)$, one has $T_{y_0}Y \subset \tau$. We say (X, Y) is (a) -regular if it is (a) -regular at any point $y_0 \in Y$.

(ii) (Bekka (c) -regularity)

Let ρ be a smooth non-negative function such that $\rho^{-1}(0) = Y$. (X, Y) is (c) -regular at $y_0 \in Y$ for the control function ρ if :

for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{\text{Ker} d\rho(x_i) \cap T_{x_i}X\}$ tends to τ in the grassmannian $G(\dim X - 1, \dim M)$, one has $T_{y_0}Y \subset \tau$. (X, Y) is (c) -regular at y_0 if it is (c) -regular for some control function ρ . We say (X, Y) is (c) -regular if it is (c) -regular at any point $y_0 \in Y$.

Let $F: (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \rightarrow (\mathbb{C}, 0)$ be a deformation of an analytic function f . We denote by $\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbb{C}, \{0\} \times \mathbb{C}\}$ the canonical stratification of the germ variety V_F of the zero locus of F . We may assume that f is convenient, this is not a restriction when it defines an isolated singularity, in fact, by adding z_i^N for a sufficiently large N for which the isomorphism class of F_t does not change. Hereafter, we will assume that f is convenient,

$$X = F^{-1}(0) - \{0\} \times \mathbb{C}, \quad Y = \{0\} \times \mathbb{C} \text{ and } \rho(z) = \sum_{\alpha \in \text{ver}(F_t)} z^\alpha \bar{z}^\alpha.$$

Here $\text{ver}(F_t)$ denotes the Newton vertices of F_t when $t \neq 0$.

Note that by the convenience assumption on f , $\rho^{-1}(0) = Y$.

We also let

$$\partial \rho = \sum_{i=1}^n \frac{\partial \rho}{\partial z_i} \frac{\partial}{\partial z_i} + \frac{\partial \rho}{\partial \bar{z}_i} \frac{\partial}{\partial \bar{z}_i} = \partial_z \rho + \partial_{\bar{z}} \rho$$

and

$$\partial F = \sum_{i=1}^n \frac{\partial F}{\partial z_i} \frac{\partial}{\partial z_i} + \frac{\partial F}{\partial t} = \partial_z F + \partial_t F.$$

Calculation of the map $\partial_z \rho|_X$

First of all we remark that $\partial_z \rho = \partial_z \rho|_X + \partial_z \rho|_N$ (where N denotes the normal space to X). Since N is generated by the gradient of F , we have that $\partial_z \rho = \partial_z \rho|_X + \eta \partial F$. On the other hand, $\langle \partial_z \rho|_X, \partial F \rangle = 0$, so we get $\eta = \frac{\langle \partial_z \rho, \partial F \rangle}{|\partial F|^2}$. It follows that

$$(2.1) \quad \partial_z \rho|_X = \partial_z \rho - \frac{\langle \partial_z \rho, \partial F \rangle}{|\partial F|^2} \partial F = (\partial_z \rho|_X)_z + (\partial_z \rho|_X)_t,$$

we have

$$(\partial_z \rho|_X)_z = \partial_z \rho - \frac{\langle \partial_z \rho, \partial F \rangle}{|\partial F|^2} \partial_z F, \quad (\partial_z \rho|_X)_t = -\frac{\langle \partial_z \rho, \partial F \rangle}{|\partial F|^2} \partial_t F$$

and

$$|\partial_z \rho|_X|^2 = \frac{|\partial F|^2 |\partial_z \rho|^2 - |\langle \partial_z \rho, \partial F \rangle|^2}{|\partial F|^2} = \frac{\|\partial F \wedge \partial_z \rho\|^2}{|\partial F|^2}.$$

Then we can characterise the (c) -regularity as follows :

Theorem 2.1. *Consider X and Y as above. The following conditions are equivalent*

- (i) (X, Y) is (c)-regular for the control function ρ .
- (ii) (X, Y) is (a)-regular and $|(\partial_z \rho|_X)_t| \ll |\partial_z \rho|_X|$ as $(z, t) \in X$ and $(z, t) \rightarrow Y$.
- (iii) $|\partial_t F| \ll \frac{\|\partial F \wedge \partial_z \rho\|}{|\partial_z \rho|}$ as $(z, t) \in X$ and $(z, t) \rightarrow Y$.

Proof. Since (i) \Leftrightarrow (ii) is proved in ([1], Theorem 1), and (iii) \Rightarrow (ii) is trivial, it is enough to see (ii) \Rightarrow (iii).

To show that (ii) \Rightarrow (iii), it suffices to show this on any analytic curve $\lambda(s) = (z(s), t(s)) \in X$ and $\lambda(s) \rightarrow Y$. Indeed, we have to distinguish two cases :

First case, we suppose that along λ , $|\langle \partial_z \rho, \partial F \rangle| \sim |\partial_z \rho| |\partial F|$, hence by (2.1) and (ii), we have

$$|(\partial_z \rho|_X)_t| = \frac{|\langle \partial_z \rho, \partial F \rangle|}{|\partial F|^2} |\partial_t F| \ll \frac{\|\partial F \wedge \partial_z \rho\|}{|\partial F|}.$$

But this clearly implies

$$|\partial_t F| \ll \frac{\|\partial F \wedge \partial_z \rho\|}{|\partial_z \rho|} \quad \text{along the curve } \lambda(s),$$

where $|\langle \partial_z \rho, \partial F \rangle| \sim |\partial_z \rho| |\partial F|$.

Second case, we suppose that along λ , $|\langle \partial_z \rho, \partial F \rangle| \ll |\partial_z \rho| |\partial F|$, thus

$$\|\partial F \wedge \partial_z \rho\| \sim |\partial_z \rho| |\partial F| \quad \text{along the curve } \lambda(s).$$

On the other hand, by the Whitney (a)-regularity in (ii) we get

$$|\partial_t F| \ll |\partial F|.$$

Therefore, $|\partial_t F| \ll |\partial F| \sim \frac{\|\partial F \wedge \partial_z \rho\|}{|\partial_z \rho|}$ along the curve $\lambda(s)$. The Theorem 2.1 is proved \square

3. PROOF OF THE THEOREM 1.1

Before starting the proofs, we will recall some important results on the Newton number and the geometric characterization of μ -constancy.

Theorem 3.1 (A. G. Kouchnirenko [7]). *Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated singularity, then the Milnor number $\mu(f) \geq \nu(f)$. Moreover, the equality holds if f is non-degenerate.*

As an immediate corollary we have

Corollary 3.2 (M. Furuya [6]). *Let $f, g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions with $\Gamma_+(g) \subset \Gamma_+(f)$. Then $\nu(g) \geq \nu(f)$.*

On the other hand, concerning the μ -constancy, we have

Theorem 3.3 (Greuel [5], Lê-Saito [10], Teissier [16]). *Let $F: (\mathbb{C}^n \times \mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ be the deformation of a holomorphic $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with isolated singularity. The following statements are equivalent.*

- (1) F is a μ -constant deformation of f .
- (2) $\frac{\partial F}{\partial t_j} \in \overline{J(F_t)}$, where $\overline{J(F_t)}$ denotes the integral closure of the Jacobian ideal of F_t generated by the partial derivatives of F with respect to the variables z_1, \dots, z_n .

(3) The deformation $F(z, t) = F_t(z)$ is a Thom map, that is,

$$\sum_{j=1}^m \left| \frac{\partial F}{\partial t_j} \right| \ll \|\partial F\| \text{ as } (z, t) \rightarrow (0, 0).$$

(4) The polar curve of F with respect to $\{t = 0\}$ does not split, that is,

$$\{(z, t) \in \mathbb{C}^n \times \mathbb{C}^m \mid \partial_z F(z, t) = 0\} = \{0\} \times \mathbb{C}^m \text{ near } (0, 0).$$

We now want to prove theorem 1.1, in fact, let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be a deformation of a holomorphic germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that F_t is non-degenerate. Then, by theorem 3.1, we have

$$(3.1) \quad \mu(f) = \nu(f) = \mu(F_t) = \nu(F_t).$$

Consider the deformation \tilde{F} of f given by

$$\tilde{F}(z, t, \lambda) = F_t(z) + \sum_{\alpha \in \text{ver}(F_t)} \lambda_\alpha z^\alpha.$$

From the upper semi-continuity of Milnor number [11], we obtain

$$(3.2) \quad \mu(f) \geq \mu(\tilde{F}_{t, \lambda}) \text{ for } (t, \lambda) \text{ near } (0, 0).$$

By Theorem 3.1 and Corollary 3.2 therefore

$$\mu(\tilde{F}_{t, \lambda}) \geq \nu(\tilde{F}_{t, \lambda}) \geq \nu(F_t).$$

It follows from (3.1) and (3.2) that the deformation \tilde{F} is μ -constant, and hence, by Theorem 3.3 we get

$$(3.3) \quad |\partial_t F| + \sum_{\alpha \in \text{ver}(F_t)} |z^\alpha| \ll |\partial_z F + \sum_{\alpha \in \text{ver}(F_t)} \lambda_\alpha z^\alpha| \text{ as } (z, t, \lambda) \rightarrow (0, 0, 0).$$

Therefore, for all $\alpha \in \text{ver}(F_t)$ we have $|z^\alpha| \ll |\partial_z f|$, and so $m(z^\alpha) \geq m(f)$. Hence the equality $m(F_t) = m(f)$ follows. In other words, F is equimultiple.

We also show that condition (3.3), in fact, implies Bekka's (c)-regularity, hence, this deformation is topologically trivial. For this purpose, we need the following lemma (see [15]).

Lemma 3.4. *Suppose F_t is a deformation as above, then we have*

$$(3.4) \quad \sum_{\alpha \in \text{ver}(F_t)} |z^\alpha| \ll \inf_{\eta \in \mathbb{C}} \{|\partial F + \sum_{\alpha \in \text{ver}(F_t)} \eta \bar{z}^\alpha \partial_z z^\alpha|\} \text{ as } (z, t) \rightarrow (0, 0), F(z, t) = 0.$$

Proof. Suppose (3.4) does not hold. Then by the curve selection lemma, there exists an analytic curve $p(s) = (z(s), t(s))$ and an analytic function $\eta(s)$, $s \in [0, \epsilon)$, such that :

- (a) $p(0) = 0$;
- (b) $F(p(s)) \equiv 0$, and hence $dF(p(s)) \frac{dp}{ds} \equiv 0$;
- (c) along the curve $p(s)$ we have

$$\sum_{\alpha \in \text{ver}(F_t)} |z^\alpha| \gtrsim |\partial F + \sum_{\alpha \in \text{ver}(F_t)} \eta(s) \bar{z}^\alpha \partial_z z^\alpha|.$$

Set

$$(3.5) \quad g(z, \bar{z}) = \left(\sum_{\alpha \in \text{ver}(F_t)} \bar{z}^\alpha z^\alpha \right)^{\frac{1}{2}} \quad \text{and} \quad \gamma(s) = \eta(s)g(z(s), \bar{z}(s))$$

First suppose that $\gamma(s) \rightarrow 0$. Since $|\bar{z}^\alpha| \leq g$, we have,

$$\lambda_\alpha = \frac{\gamma(s)\bar{z}^\alpha(s)}{g(z(s), \bar{z}(s))} \rightarrow 0, \quad \forall \alpha \in \text{ver}(F_t)$$

Next, using (3.3) and (3.5) it follows

$$\sum_{\alpha \in \text{ver}(F_t)} |z^\alpha(s)| \ll |\partial F(p(s)) + \sum_{\alpha \in \text{ver}(F_t)} \eta(s)\bar{z}^\alpha(s) \partial_z z^\alpha(s)| \quad \text{as } s \rightarrow 0,$$

which contradicts (c).

Suppose now that the limit of $\gamma(s)$ is not zero (i.e., $|\gamma(s)| \gtrsim 1$). Since $p(0) = 0$ and $g(z(0), \bar{z}(0)) = 0$, we have, asymptotically as $s \rightarrow 0$,

$$(3.6) \quad s \left| \frac{dp}{ds}(s) \right| \sim |p(s)| \quad \text{and} \quad s \frac{d}{ds} g(z(s), \bar{z}(s)) \sim g(z(s), \bar{z}(s)).$$

But

$$(3.7) \quad \frac{d}{ds} g(z(s), \bar{z}(s)) = \sum_{\alpha \in \text{ver}(F_t)} \frac{1}{2g(z(s), \bar{z}(s))} \left(\bar{z}^\alpha dz^\alpha \frac{dz}{ds} + z^\alpha d\bar{z}^\alpha \frac{d\bar{z}}{ds} \right).$$

We have $\bar{z}^\alpha dz^\alpha \frac{dz}{ds} = \overline{z^\alpha d\bar{z}^\alpha \frac{d\bar{z}}{ds}}$ and $1 \lesssim |\gamma(s)|$. Thus,

$$(3.8) \quad \left| \frac{d}{ds} g(z(s), \bar{z}(s)) \right| \lesssim \left| \sum_{\alpha \in \text{ver}(F_t)} \frac{\gamma(s)}{g(z(s), \bar{z}(s))} \bar{z}^\alpha dz^\alpha \frac{dz}{ds} \right|.$$

This together with (3.6), (3.5) and (b) gives

$$g(z(s), \bar{z}(s)) \sim \left| s \frac{d}{ds} g(z(s), \bar{z}(s)) \right| \lesssim s \left| \sum_{\alpha \in \text{ver}(F_t)} \eta(s) \bar{z}^\alpha dz^\alpha \frac{dz}{ds} + dF(p(s)) \frac{dp}{ds} \right|.$$

Hence

$$g(z(s), \bar{z}(s)) \lesssim s \left| \frac{dp}{ds}(s) \right| \left| \sum_{\alpha \in \text{ver}(F_t)} \eta(s) \bar{z}^\alpha \partial z^\alpha + \partial F(p(s)) \right|,$$

which contradicts (c). This ends the proof of Lemma. \square

We shall complete the proof of Theorem 1.1 Since $\Gamma_+(\partial_t F) \subset \Gamma_+(F_t)$. Then, by an argument, based again on the curve selection lemma, we get the following inequality

$$(3.9) \quad |\partial_t F| \lesssim \sum_{\alpha \in \text{ver}(F_t)} |z^\alpha|.$$

Then, by the above Lemma 3.4, we obtain

$$|\partial_t F| \ll \inf_{\eta \in \mathbb{C}} \{ |\partial F + \eta \partial_z \rho| \} \quad \text{as } (z, t) \rightarrow (0, 0), \quad F(z, t) = 0,$$

we recall that

$$\rho(z) = \sum_{\alpha \in \text{ver}(F_t)} z^\alpha \bar{z}^\alpha.$$

But

$$\inf_{\eta \in \mathbb{C}} \{|\partial F + \eta \partial_z \rho|^2\} = \frac{|\partial F|^2 |\partial_z \rho|^2 - |\langle \partial_z \rho, \partial F \rangle|^2}{|\partial_z \rho|^2} = \frac{\|\partial F \wedge \partial_z \rho\|^2}{|\partial_z \rho|^2}.$$

Therefore, by Theorem 2.1, we see that the canonical stratification $\Sigma(V_F)$ is (c)-regular for the control function ρ , then F is a topologically trivial deformation (see[3]).

This completes the proof of Theorem 1.1.

Remark 3.5. We should mention that our arguments still hold for any μ -constant deformation F of a weighted homogeneous polynomial f with isolated singularity. Indeed, we can find from Varchenko's theorem [18] that $\mu(f) = \nu(f) = \mu(F_t) = \nu(F_t)$. Thus, the above proof can be applied.

Unfortunately this approach does not work, if we only suppose that f is non-degenerate. For consider the example of Altman [2] defined by

$$F_t(x, y, z) = x^5 + y^6 + z^5 + y^3 z^2 + 2tx^2 y^2 z + t^2 x^4 y,$$

which is a μ -constant degenerate deformation of the non-degenerate polynomial $f(x, y, z) = x^5 + y^6 + z^5 + y^3 z^2$. He showed that this family has a weak simultaneous resolution. Thus, by Laufer's theorem [8], F is a topologically trivial deformation. But we cannot apply the above proof because $\mu(f) = \nu(f) = \mu(F_t) = 68$ and $\nu(F_t) = 67$ for $t \neq 0$.

We conclude with several examples.

Example 3.6. Consider the family given by

$$F_t(x, y, z) = x^{13} + y^{20} + zx^6 y^5 + tx^6 y^8 + t^2 x^{10} y^3 + z^l, \quad l \geq 7.$$

It is not hard to see that this family is non-degenerate. Moreover, by using the formula for the computation of Newton number we get $\mu(F_t) = \nu(F_t) = 153l + 32$. Thus, by theorem 1.1, we have that F_t is topologically trivial. We remark that this deformation is not μ^* -constant, in fact, the Milnor numbers of the generic hyperplane sections $\{z = 0\}$ of F_0 and F_t for $t \neq 0$ are 260 and 189 respectively.

Example 3.7. Let

$$F_t(x, y, z) = x^{10} + x^3 y^4 z + y^l + z^l + t^3 x^4 y^5 + t^5 x^4 y^5$$

where $l \geq 6$. Since $\mu(F_t) = 2l^2 + 32l + 9$ and F_t is a non-degenerate family, it follows from Theorem 1.1 that F is a topologically trivial deformation

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DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DES SCIENCES, DE TECHNOLOGIE ET DE MDECINE BP.
 880, NOUAKCHOTT, MAURITANIE
E-mail address: ymoine@univ-nkc.mr